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Nonequilibrium Potentials Near Instabilities

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The existence of polynomial approximations for nonequilibrium potentials determined by a master equation near an instability of arbitrary codimension with diagonalizable linear part is studied. It is shown that the approximations exist, provided some relations are satisfied between the coefficients of the master equation.

KEY WORDS: Bifurcations; dynamical systems; instabilities; master equation; Markov processes; nonequilibrium potentials.

1. INTRODUCTION

The practical determination of a nonequilibrium potential associated with a Markov process has received considerable attention recently. In the case of diffusion processes obeying a Fokker–Planck equation, Graham and collaborators have shown that the potential is generically nondifferentiable.[1] It has, however, been shown by these authors that in the neighborhood of codimension-one instabilities the potential admit a polynomial approximation and that this also holds in the neighborhood of codimension-two bifurcations with diagonalizable linear part for a special choice of noise sources.[2] In fact, in this last reference a much more general result is obtained, since near codimension-two bifurcations smooth nonpolynomial potentials are explicitly constructed which reduce to polynomial form in special cases. The problem of polynomial expansions has also been considered recently in ref. 16 for two-variable systems and in the case of Markov processes obeying a canonical master equation it has been studied by Lemarchand and Nicolis.[3] In particular, Lemarchand and collaborators[4,12] have recovered the polynomial expansions of Graham

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for the codimension-two instabilities $(\Omega_1 \Omega_2)$ and $(\xi\Omega)$ provided that a relation is valid between some coefficients of the master equation, a condition which is equivalent to the special choice of noise sources in ref. 2. We shall study here the general problem of the conditions that the coefficients of the master equation must satisfy in order that the potential admits, near an arbitrary instability with diagonalizable linear part, an approximation with polynomial dependence in the gross variables and in the unfolding parameters.

Our starting point is a master equation involving a small expansion parameter $\eta = \Omega^{-1}$ in canonical form in the sense of van Kampen.[5] This means that if $Q = (Q_1, Q_2,..., Q_N)$ are the gross variables, the transition probability is of the form $(q_\mu = \Omega^{-1} Q_\mu,\ r_\mu = Q_\mu - Q'_\mu,\ f(\Omega)$ a given function)

$$W(Q \mid Q') = f(\Omega) \sum_{\alpha \geqslant 0} \eta^\alpha w_\alpha(q, r) \tag{1}$$

and then after a scaling of time $t' = \eta f(\Omega)\, t$ the master equation takes the form $(\partial_\mu = \partial/\partial q_\mu)$

$$\eta \frac{\partial}{\partial t} p(q, t) = \sum_{\alpha \geqslant 0} \eta^\alpha \sum_r \left[\exp\left(-\eta \sum_\mu r_\mu \partial_\mu \right) - 1 \right] \times w_\alpha(q, r)\, p(q, t) \tag{2}$$

which can be written as

$$\eta \frac{\partial}{\partial t} p(q, t) = \sum_{\alpha \geqslant 0} \eta^\alpha L_\alpha(q, \eta\nabla)\, p(q, t) \tag{3}$$

with

$$L_\alpha(q, \eta\nabla) = \sum_{n \geqslant 1} \eta^n \partial_{\mu_1} \cdots \partial_{\mu_n} A_\alpha^{\mu_1 \cdots \mu_n}(q) \tag{4}$$

Following Kubo *et al.*,[6] we can associate a problem of classical mechanics with this master equation defining the Hamiltonian

$$H(q, p) = L_0(q, \eta\nabla = -p) = \sum_{n \geqslant 1} (-1)^n p_{\mu_1} \cdots p_{\mu_n} A^{\mu_1 \cdots \mu_n}(q) \tag{5}$$

In the Fokker–Planck case η measures the intensity of the noise sources and here of we call Ω the volume (which is often the case), it represents the inverse volume. We shall be interested in the limit $\eta \to 0$, for which a WKB-type expansion can be set up, which gives the transition probability density in the form[7]

$$P(q, t \mid q_0, t_0) = P_{\mathrm{WKB}}(q, t \mid q_0, t_0)\Big(I_0 + \sum_{n \geqslant 1} \eta^n I_n\Big) \tag{6}$$

where

$$P_{\mathrm{WKB}}(q, t \mid q_0, t_0) = \exp\left[-\frac{1}{\eta} A(q, t \mid q_0, t_0)\right] \tag{7}$$

with $A(q, t \mid q_0, t_0)$ the classical action for the specified boundary conditions calculated with the Hamiltonian (5). Using functional integral techniques, one can give explicit formulas for all the corrections I_j in (6) in terms of the solution of the mechanical problem defined by (5).[7,8] The form (7) shows that we can write a WKB approximation $p^{\mathrm{st}}_{\mathrm{WKB}}(q) \approx \exp[-(1/\eta)\,\phi(q)]$ for the stationary probability, where $\phi(q)$ is called the potential, which satisfies then the Hamilton–Jacobi equation $H(q, \nabla\phi) = 0$. Putting

$$H_{\mu_1 \cdots \mu_n}(q) = n!\,(-1)^n A_0^{\mu_1 \cdots \mu_n}(q)$$

we remark that in the limit $\eta \to 0$ the master equation would reduce to a deterministic one and the deterministic macroscopic equations would be $\dot{q}_\alpha = H_\alpha(q)$. We shall speak of a bifurcation here when the vector field $H_\alpha(q)$ [we assume $H_\alpha(0) = 0$, i.e., $q = 0$ is an equilibrium] becomes singular, i.e., its linear part has eigenvalues with vanishing real part. We treat an instability with diagonalizable linear part and such that the equilibrium $q = 0$ is persistent in a neighborhood of the critical point (the point where $q = 0$ loses its stability) in the space of parameters. Then, if we assume that by a nonlinear change of variables, $H_\alpha(q)$ has been put in normal form,[9] its linear part will be diagonal and at the critical point for the $(m\xi)(\Omega_1 \cdots \Omega_p)$ instability it will have eigenvalues $\{\sigma_\alpha, 1 \leqslant \alpha \leqslant N\}$, where $\sigma_\alpha = 0$, $1 \leqslant \alpha \leqslant m$; $\sigma_{m+2j-1} = i\Omega_j$, $\sigma_{m+2j} = -i\Omega_j$, $1 \leqslant j \leqslant p$; $\sigma_{n+\alpha} = \gamma_\alpha$, $1 \leqslant \alpha \leqslant N - n$, $\operatorname{Re} \gamma_\alpha < 0$; and $n = m + 2p$ is the dimension of the critical space. It can be shown that the stable variables corresponding to the eigenvalues $\{\gamma_\alpha\}$ can be eliminated from the problem (for the Fokker–Planck case see ref. 10; the result is the same for the master equation). Finally, we have to solve a Hamilton–Jacobi equation $H(q_1,..., q_n, \partial_1\phi,..., \partial_n\phi) = 0$ in the critical variables. Putting

$$H(q, p) = \sum (r!)^{-1} H_{\mu_1} \cdots \mu_r(q)\, p_{\mu_1} \cdots p_{\mu_r}$$

the vector field $\{H_\mu(q_1,..., q_m, z_1, \bar{z}_1,..., z_p, \bar{z}_p), 1 \leqslant \mu \leqslant n\}$ (here $q_{m+2j-1} = z_j$, $q_{m+2j} = \bar{z}_j$, $1 \leqslant j \leqslant p$, and $\bar{z}_j$ is the complex conjugate of z_j) has the following normal form in the unfolding of the singularity[9]:

$$H_\alpha(q) = \mu_\alpha q_\alpha + F_\alpha(q_1,..., q_m, |z_1|^2,..., |z_p|^2), \qquad 1 \leqslant \alpha \leqslant m \quad \text{(8a)}$$

$$H_{m+2\alpha-1}(q) = (\mu_{m+\alpha} + i\Omega_\alpha) z_\alpha + z_\alpha G_\alpha(q_1,..., q_m, |z_1|^2,..., |z_p|^2) \quad \text{(8b)}$$

where $1 \leqslant \alpha \leqslant p$, and $H_{m+2\alpha}(q)$ is the complex conjugate of (8b). In (8), $\{\mu_\alpha, 1 \leqslant \alpha \leqslant m+p\}$ are the unfolding parameters (the instability is of codimension $m+p$), the critical point is $\{\mu_\alpha = 0, 1 \leqslant \alpha \leqslant m+p\}$, and we have assumed that the frequencies $\{\Omega_\alpha, 1 \leqslant \alpha \leqslant p\}$ are nonresonant. We remark here that we have introduced in the standard way complex variables to write the normal form of $H_\mu(q)$. Originally the master equation is real and if the real critical variables are $(q'_1,..., q'_n)$, one has to solve a Hamilton–Jacobi equation $H'(q', \nabla'\phi) = 0$. One puts $q_j = q'_j$, $1 \leqslant j \leqslant m$, and $q_{m+2k-1} = z_k = q'_{m+2k-1} + iq'_{m+2k}$, $q_{m+2k} = \bar{z}_k = q'_{m+2k-1} - iq'_{m+2k}$, $1 \leqslant k \leqslant p$, and this gives the normal form (8) corresponding to the Hamiltonian $H(q, \nabla\phi) = H'(q', \nabla'\phi)$. The matrix

$$Q'_{\alpha\beta} = \left. \frac{\partial^2 H'(q', \nabla'\phi)}{\partial(\partial'_\alpha \phi)\, \partial(\partial'_\beta \phi)} \right|_{q'=0}$$

must be positive definite and this implies restrictions on the matrix

$$Q_{\alpha\beta} = \left. \frac{\partial^2 H(q, \nabla\phi)}{\partial(\partial_\alpha \phi)\, \partial(\partial_\beta \phi)} \right|_{q=0}$$

2. SOLUTION OF THE HAMILTON–JACOBI EQUATION

In order to solve $H(q, \nabla\phi) = 0$, we put $\phi = \sum_{r \geqslant 2} \phi^{[r]}$, where $\phi^{[r]}(q)$ is of polynomial order r in $(q_1 \cdots q_n)$ and develop $H(q, \nabla\phi) = 0$ in powers of q. The notation $(\cdots)^{[r]}$ stands for the terms in $(\cdots)$ which are of order r in q. We put

$$H_\mu(q) = \sum_{\nu=1}^{n} B_{\mu\nu} q_\nu + O(q^2), \qquad Q_{\mu\nu} = H_{\mu\nu}(q=0)$$

The sum for ϕ starts with $\phi^{[2]}$, since $q = 0$ is an extremum of the potential ϕ and one must impose $\nabla\phi = 0$ there.[1] Our set of equations is then $H(q, \nabla\phi)^{[r]} = 0$, $r \geqslant 2$.

One has

$$H(q, \nabla\phi)^{[r]} = \sum_{s\geqslant 1} \sum_{j_0 \cdots j_s} \frac{1}{s!} H_{\mu_1\mu_2\cdots\mu_s}(q)^{[j_0]} (\partial_{\mu_1}\phi)^{[j_1]} \times \cdots (\partial_{\mu_s}\phi)^{[j_s]} \tag{9}$$

with $j_0 \geqslant 0$, $j_k \geqslant 1$ for $k \geqslant 1$, $\sum_{k=0}^{s} j_k = r$.

Putting $\phi^{[2]} = \frac{1}{2}\sum_{\mu,\nu=1}^{n} \Lambda_{\mu\nu} q_\mu q_\nu$, one obtains from (9) for $r = 2$ the equation (B^T is the transposed matrix of B)

$$\Lambda B + B^T \Lambda + \Lambda Q \Lambda = 0 \tag{10}$$

The diagonal matrix is given by $B = A + \sum_{j=1}^{m+p} \mu_j B^{(j)}$, with

$$A_{\mu\nu} = \sum_{j=1}^{p} i\Omega_j(\delta_{\mu,m+2j-1}\delta_{\nu,m+2j-1} - \delta_{\mu,m+2j}\delta_{\nu,m+2j}) \tag{11a}$$

$$B_{\mu\nu}^{(j)} = \delta_{\mu,j}\delta_{\nu,j}, \qquad 1 \leqslant j \leqslant m \tag{12a}$$

$$B_{\mu\nu}^{(m+k)} = \delta_{\mu,m+2k-1}\delta_{\nu,m+2k-1} + \delta_{\mu,m+2k}\delta_{\nu,m+2k}, \qquad 1 \leqslant k \leqslant p \tag{12b}$$

It is simple to see that (10) implies that Λ vanishes when $\mu_j \to 0$, and moreover that it has the expansion

$$\Lambda = \sum_{j=1}^{m+p} \mu_j \Lambda^{(j)} + O(\mu_k \mu_l) \qquad \text{if} \quad Q_{ij} = Q_{ii}\delta_{ij}$$

($i \leqslant m$, $j \leqslant m$). This condition is the first relation we need on the coefficients of the master equation in order to have a polynomial dependence in the unfolding parameters for the potential. We assume that it is satisfied from now on and we shall come back to its interpretation at the end of this section. Then we have

$$\Lambda_{\mu\nu}^{(j)} = -\frac{2}{Q_{jj}} \delta_{\mu,j}\delta_{\nu,j}, \qquad 1 \leqslant j \leqslant m \tag{13a}$$

$$\Lambda_{\mu\nu}^{(m+k)} = -\frac{2}{Q_{m+2k-1,m+2k}} (\delta_{\mu,m+2k-1}\delta_{\nu,m+2k} + \delta_{\mu,m+2k}\delta_{\nu,m+2k-1}), \qquad 1 \leqslant k \leqslant p \tag{13b}$$

We remark that Q_{jj}, $1 \leqslant j \leqslant m$, and $Q_{m+2k-1,m+2k}$, $1 \leqslant k \leqslant p$, are real and positive. One obtains from (9) for $r \geqslant 3$ equations of the form

$$L\phi^{[r]} = I^{[r]}, \qquad L = \sum_{\mu,\nu=1}^{n} L_{\mu\nu} q_\nu \frac{\partial}{\partial q_\mu} \tag{14}$$

with $L = B + QA$ and where $I^{[r]}$ depends only on $\{\phi^{[s]}, s < r\}$. We see then that we can try to solve (14) by recursion in r. But we are interested in obtaining solutions of (14) with polynomial dependence on the unfolding parameters $\{\mu_j, 1 \leqslant j \leqslant m+p\}$. We put then

$$\phi^{[r]} = \phi_0^{[r]} + \sum_{j=1}^{m+p} \mu_j \phi_j^{[r]} + O(\mu_j \mu_k)$$

$$I^{[r]} = I_0^{[r]} + \sum_{j=1}^{m+p} \mu_j I_j^{[r]} + O(\mu_j \mu_k)$$

One also has $L = L^{(0)} + \sum_{j=1}^{m+p} \mu_j L^{(j)} + O(\mu_j \mu_k)$, with

$L^{(0)} = A$, $L^{(j)} = B^{(j)} + QA^{(j)}$, $1 \leqslant j \leqslant m+p$, and $\mathscr{L} = \mathscr{L}^{(0)} + \sum_{j=1}^{m+p} \mu_j \mathscr{L}^{(j)}$ with $L_{(0)}^{(j)}$ defined as in (14) with L replaced by $L^{(j)}$. Putting $D = L^{(0)}$, we obtain for the $\{\phi_j^{[r]}, 0 \leqslant j \leqslant m+p\}$ the equations

$$D\phi_0^{[r]} = I_0^{[r]} \tag{15a}$$

$$D\phi_j^{[r]} = -L^{(j)}\phi_0^{[r]} + I_j^{[r]} \equiv J_j^{[r]}, \qquad 1 \leqslant j \leqslant m+p \tag{15b}$$

We have to solve these equations starting with (15a) at each step of the recursion in r (note that $I_j^{[r]}$ depends only on $\phi^{[s]}$, $s < r$). Using (11a), one obtains

$$D = \sum_{j=1}^{p} i\Omega_j \left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right) \tag{16}$$

From (12) and (13) one gets

$$L_{\mu\nu}^{(j)} = -\delta_{\mu j}\delta_{\nu j} - 2\delta_{\nu j} \frac{Q_{\mu j}}{Q_{jj}} (1 - \delta_{\mu j}), \qquad 1 \leqslant j \leqslant m \tag{17a}$$

$$\begin{aligned} L_{\mu\nu}^{(m+j)} = &-\delta_{\mu, m+2j-1}\delta_{\nu, m+2j-1} - \delta_{\mu, m+2j}\delta_{\nu, m+2j} \\ &- 2\delta_{\nu, m+2j-1} \frac{Q_{\mu, m+2j}}{Q_{m+2j-1, m+2j}} (1 - \delta_{\mu, m+2j-1}) \\ &- 2\delta_{\nu, m+2j} \frac{Q_{\mu, m+2j-1}}{Q_{m+2j-1, m+2j}} (1 - \delta_{\mu, m+2j}), \qquad 1 \leqslant j \leqslant p \end{aligned} \tag{17b}$$

and the $L^{(j)}$ are of the form

$$L^{(j)} = q_j \left(-\frac{\partial}{\partial q_j} + \sum_{\mu \neq j} L_{\mu j}^{(j)} \frac{\partial}{\partial q_\mu} \right), \qquad 1 \leqslant j \leqslant m \tag{18a}$$

$$L^{(m+j)} = z_j \left(-\frac{\partial}{\partial z_j} + \sum_{\mu \neq m+2j-1} L^{(m+j)}_{\mu, m+2, j-1} \frac{\partial}{\partial q_\mu} \right)$$
$$+ \bar{z}_j \left(-\frac{\partial}{\partial \bar{z}_j} + \sum_{\mu \neq m+2j} L^{(m+j)}_{\mu, m+2j} \frac{\partial}{\partial q_\mu} \right), \qquad 1 \leqslant j \leqslant p \tag{18b}$$

Before solving Eqs. (15), we need some preliminaries. Let $\mathscr{H}$ be the space of formal power series in $(q_1, q_2,..., q_n)$ and $\mathscr{H}^{(r)}$ the subspace of $\mathscr{H}$ of polynomials of degree r in $(q_1,..., q_n)$. The operators L, D, and $L^{(j)}$ act on $\mathscr{H}$ and leave $\mathscr{H}^{(r)}$ invariant for all r and in general if an operator R acting in $\mathscr{H}$ leaves the spaces $\mathscr{H}^{(r)}$ invariant, we shall write $R \mid \mathscr{H}^{(r)}$ for the restriction of R to $\mathscr{H}^{(r)}$. We define now a suitable scalar product $\langle \cdot, \cdot \rangle$ (antilinear in the first argument and linear in the second) in $\mathscr{H}$ which is such that the monomials

$$\left\{ \left(\prod_{j=1}^{n} m_j! \right)^{-1/2} q_1^{m_1} q_2^{m_2} \cdots q_n^{m_n}, m_j \text{ entire numbers} \right\}$$

form an orthonormal basis[11] (see also ref. 9). With this scalar product the adjoint of the operator q_j (multiplication by q_j) is simply $\partial/\partial q_j$, i.e., $q_j^+ = \partial/\partial q_j$, and consequently $D^+ = -D$. Then $\operatorname{Ker} D = \{ f \in \mathscr{H}: Df = 0 \} = \operatorname{Ker} D^+$ and it is easy to see that it is the space of functions $F(q_1,..., q_m, |z_1|^2,..., |z_p|^2)$, which can be developed in formal power series in their arguments $(q_1,..., q_m, |z_1|^2,..., |z_p|^2)$.

We come back now to the solutions of (15) which are equations in $\mathscr{H}^{(r)}$. In order to solve (15a) for $\phi_0^{[r]}$, one must impose that $I_0^{[r]}$ is orthogonal to $\operatorname{Ker} D^+ \upharpoonright \mathscr{H}^{(r)} = \operatorname{Ker} D \upharpoonright \mathscr{H}^{(r)}$ (Fredholm alternative) and this imposes relations between the coefficients of the $\{\phi_0^{[s]}, s < r\}$ and of the master equation. Once this solvability condition $I_0^{[r]} \perp \operatorname{Ker} D^+ \upharpoonright \mathscr{H}^{(r)}$ is satisfied, we can solve (15a) in the form $\phi_0^{[r]} = \chi_0^{[r]} + \psi_0^{[r]}$, where $\chi_0^{[r]}$ is a particular solution of (15a) and $\psi_0^{[r]} \in \operatorname{Ker} D$ is of the form

$$\psi_0^{[r]} = \sum_{j_k \geqslant 0} a^{(r)}_{0, j_1} \cdots j_{m+p} q_1^{j_1} \cdots q_m^{j_m} |z_1|^{2j_{m+1}} \cdots |z_p|^{2j_{m+p}} \tag{19}$$

with $\sum_{i=1}^{m} j_i + 2 \sum_{i=1}^{p} j_{m+i} = r$ and the $\{a^{(r)}_{0, j_1 \cdots j_{m+p}}\}$ are arbitrary coefficients. One has the form (19), since the monomials

$$\left\{ q_1^{j_1} \cdots q_m^{j_m} |z_1|^{2j_{m+1}} \cdots |z_p|^{2j_{m+p}}, \sum_{k=1}^{m} j_k + 2 \sum_{k=1}^{p} j_{m+k} = r \right\} \tag{20}$$

which we call the resonant terms, form a basis of the space $\operatorname{Ker} D \upharpoonright \mathscr{H}^{(r)}$. We remark that D is closely related to the homological operator $A = D - A$ associated to the instability (see ref. 9. We replace now $\phi_0^{[r]}$ in (15b) and we must impose $J_j^{[r]} \perp \operatorname{Ker} D^+ \upharpoonright \mathscr{H}^{(r)}$, $1 \leqslant j \leqslant m+p$. This gives a set of relations linear in the $\{a^{(r)}_{0,j_1 \cdots j_{m+p}}\}$, which overdetermines them in the sense that one has more relations than coefficients and this leads again to relations among the coefficients of the master equation. This general mechanism of solvability conditions via the Fredholm alternative is then what produces the relations found in refs. 2, 4, and 12, which are then a consequence of imposing a polynomial expansion in $(q_1 \cdots q_n)$ and in the unfolding parameters $(\mu_1 \cdots \mu_{m+p})$.

We shall obtain now explicitly the first nontrivial order from (15). Let the entire number $r' \geqslant 3$ be such that the first nonlinearity in the normal form of the vector field $H_\mu(q)$ arrives at $s = r' - 1$, i.e., one has

$$H_\mu(q) = H_\mu(q)^{[1]} + \sum_{s \geqslant r'-1} H_\mu(q)^{[s]}$$

In the generic case $r' = 3$ unless there is a symmetry in the problem. For example, the symmetry $q_j \to -q_j$, $1 \leqslant j \leqslant m$, implies $r' = 4$ and also guarantees the persistence of the equilibrium at the origin $q = 0$, which we are assuming here. It is easy to check from the expansion (9) that $I_0^{[s]} = 0$ for $s < r'$ and this will imply, as we shall see, $I_j^{[s]} = 0$, $1 \leqslant j \leqslant m+p$, for $s < r'$. We suppose $r' > 3$; then Eqs. (15) for $r = 3$ are

$$D\phi_0^{[3]} = 0 \tag{21a}$$

$$D\phi_j^{[3]} = -L^{(j)}\phi_0^{[3]} \equiv J_j^{[3]}, \qquad 1 \leqslant j \leqslant m+p \tag{21b}$$

since $I_j^{[3]} = 0$ also. From (21a) we obtain $\phi_0^{[3]} \in \operatorname{Ker} D \upharpoonright \mathscr{H}^{(3)}$, i.e., it is of the form [see (19)]

$$\phi_0^{[3]} = \sum a^{(3)}_{0;j_1 \cdots j_{m+p}} q_1^{j_1} \cdots q_m^{j_m} |z_1|^{2j_{m+1}} \cdots |z_p|^{2j_{m+p}} \tag{22}$$

We replace $\phi_0^{[3]}$ in (21b); then the solvability condition $J_j^{[3]} \perp \operatorname{Ker} D^+ \upharpoonright \mathscr{H}^{(3)}$ gives, $\forall \alpha$ and $1 \leqslant j \leqslant m+p$,

$$\langle g_\alpha^{(3)}, L^{(j)}\phi_0^{[3]} \rangle = \langle L^{(j)+} g_\alpha^{(3)}, \phi_0^{[3]} \rangle = 0 \tag{23}$$

The set $\{g_\alpha^{(r)}\}$ is the basis of $\operatorname{Ker} D \upharpoonright \mathscr{H}^{(r)}$ formed by the monomials in (20). From (18) we have

$$L^{(j)+} = \left(-q_j + \sum_{\mu \neq j} \bar{L}^{(j)}_{\mu j} q_\mu \right) \frac{\partial}{\partial q_j}$$

$$= -q_j \frac{\partial}{\partial q_j} + B_1^{(j)}, \qquad 1 \leqslant j \leqslant m \tag{24a}$$

$$L^{(m+j)+} = \left(-z_j + \sum_{\mu \neq m+2j-1} \bar{L}^{(m+j)}_{\mu, m+2j-1} \, q_\mu \right) \frac{\partial}{\partial z_j}$$
$$+ \left(-\bar{z}_j + \sum_{\mu \neq m+2j} \bar{L}^{(m+j)}_{\mu, m+2j} q_\mu \right) \frac{\partial}{\partial \bar{z}_j}$$
$$= -\left(z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right) + B_1^{(m+j)}, \qquad 1 \leqslant j \leqslant p \tag{24b}$$

We can write $L^{(k)+} = B_0^{(k)} + B_1^{(k)}$, with

$$B_0^{(k)} = -q_k \frac{\partial}{\partial q_k}, \qquad 1 \leqslant k \leqslant m$$
$$B_0^{(m+k)} = -\left(z_k \frac{\partial}{\partial z_k} + \bar{z}_k \frac{\partial}{\partial \bar{z}_k} \right), \qquad 1 \leqslant k \leqslant p$$

The form of $B_1^{(k)}$ shows that $B_1^{(k)} g_\alpha^{(r)}$ is orthogonal to Ker $D \restriction \mathscr{H}^{(r)}$, while for

$$g = q_1^{j_1} \cdots q_m^{j_m} \, |z_1|^{2j_{m+1}} \cdots |z_p|^{2j_{m+p}}$$

one has

$$B_0^{(k)} g = -j_k \, g, \quad 1 \leqslant k \leqslant m; \qquad B_0^{(m+k)} g = -2j_{m+k} \, g, \quad 1 \leqslant k \leqslant p \tag{25}$$

Since $\phi_0^{[3]} \in \operatorname{Ker} D$, one obtains from (23) and (25) that $\langle g_\alpha^{(3)}, \phi_0^{[3]} \rangle = 0$, which implies that all $a^{(3)}_{0; j_1 \cdots j_{m+p}}$ in (22) vanish and $\phi_0^{[3]} = 0$. In the same way we shall have $\phi_0^{[s]} = 0$, $s < r'$. For $r = r'$ one has $I_0^{[r']} = 0$ and $I_k^{[r']}$ is given by

$$I_k^{[r']} = \frac{2}{Q_{kk}} H_k(q)^{[r'-1]} q_k, \qquad 1 \leqslant k \leqslant m \tag{26a}$$

$$I_{m+k}^{[r']} = \frac{2}{Q_{m+2k-1,\, m+2k}} \left[H_{m+2k-1}(q)^{[r'-1]} \bar{z}_k + \text{c.c.} \right], \qquad 1 \leqslant k \leqslant p \tag{26b}$$

Due to (8), we see that $I_k^{[r']}$ will contain only resonant terms. Since $I_0^{[r']} = 0$, we obtain from (15a)

$$\phi_0^{[r']} = \sum a^{(r')}_{0; j_1 \cdots j_{m+p}} q_1^{j_1} \cdots q_m^{j_m} \, |z_1|^{2j_{m+1}} \cdots |z_p|^{2j_{m+p}} \tag{27}$$

The solvability conditions of (15b) will be $(1 \leqslant k \leqslant m+p)$

$$\langle L^{(k)+} g_\alpha^{(r')}, \phi_0^{[r']} \rangle = \langle B_0^{(k)} g_\alpha^{(r')}, \phi_0^{[r']} \rangle = \langle g_\alpha^{(r')}, I_k^{[r']} \rangle \tag{28}$$

where $\{g_\alpha^{(r')}\}$ is the basis in (20). Since $I_k^{[r']}$ contain only resonant terms, i.e., $I_k^{[r']} \in \operatorname{Ker} D \upharpoonright \mathscr{H}^{(r')}$, we can write

$$I_k^{[r']} = \sum b_{k;j_1 \cdots j_{m+p}}^{(r')} q_1^{j_1} \cdots q_m^{j_m} |z_1|^{2j_{m+1}} \cdots |z_p|^{2j_{m+p}} \tag{29}$$

If

$$g_\alpha^{(r')} = q_1^{j_1} \cdots q_m^{j_m} |z_1|^{2j_{m+1}} \cdots |z_p|^{2j_{m+p}}$$

one obtains from (28) that

$$-j_k a_{0;j_1 \cdots j_{m+p}}^{(r')} = b_{k;j_1 \cdots j_{m+p}}^{(r')}, \qquad 1 \leqslant k \leqslant m \tag{30a}$$

$$-2j_{m+k} a_{0;j_1 \cdots j_{m+p}}^{(r')} = b_{m+k;j_1 \cdots j_{m+k}}^{(r')}, \qquad 1 \leqslant k \leqslant p \tag{30b}$$

If $j_k = 0$ in (30a) [respectively, if $j_{m+k} = 0$ in (30b)] $b_{k;j_1 \cdots j_{m+p}}^{(r')} = 0$, since from (26a), $I_k^{(r')}$ necessarily contains q_k as a factor [respectively $b_{m+k;j_1 \cdots j_{m+p}}^{(r')} = 0$, since from (26b) $I_k^{(r')}$ necessarily contains $|z_k|^2$ as a factor], and Eqs. (30) are identically satisfied. Let us consider now a definite coefficient $a_{0;j_1}^{(r')} \cdots j_{m+p}$ of (27) which we want to determine and let $\{i_1, i_2,..., i_q\}$, with $1 \leqslant i_1 < i_2 < \cdots < i_q \leqslant m+p$, be the subset of $\{1, 2,..., m+p\}$ such that $j_{i_l} \neq 0$, $l = 1, 2,..., q$. Then $a_{0;j_1 \cdots j_{m+p}}^{(r')}$ will appear in the left-hand side of (30) in the q equations

$$-j'_{i_l} a_{0;j_1 \cdots j_{m+p}}^{(r)} = b_{j_{i_l}}^{(r')}; j_1,..., j_{m+p}, \qquad l = 1, 2,..., q \tag{31}$$

where $j'_{i_l} = j_{i_l}$, $i_l \leqslant m$, and $j'_{i_l} = 2j_{i_l}$, $i_l > m$. This gives the $q-1$ relations

$$-\frac{1}{j'_{i_1}} b_{j_{i_1};j_1 \cdots j_{m+p}}^{(r')} = -\frac{1}{j'_{i_l}} b_{j_{i_l};j_1 \cdots j_{m+p}}^{(r')}, \qquad l = 2, 3,..., q \tag{32}$$

between the coefficients of $I_k^{(r')}$, and if they are satisfied we can determine $a_{0;j_1 \cdots j_{m+p}}^{(r')}$. Repeating the same procedure, we can determine all the coefficients of $\phi_0^{[r']}$, provided a set of relations of the type (32) are satisfied. We can now give the final result $\phi = \phi^{[2]} + \phi_0^{[r']}$ for the potential up to this order using (13) for the matrix Λ. One has

$$\begin{aligned} \phi = &-\sum_{j=1}^{n} \frac{\mu_j}{Q_{jj}} q_j^2 - 2 \sum_{j=1}^{p} \frac{\mu_{m+j}}{Q_{m+2j-1,m+2j}} |z_j|^2 \\ &+ \sum a_{0;j_1 \cdots j_{m+p}}^{(r')} q_1^{j_1} \cdots q_m^{j_m} |z_1|^{2j_{m+1}} \cdots |z_p|^{2j_{m+p}} \\ &+ O(q_k^{r'+1}, \mu_j q_k^3) \end{aligned} \tag{33}$$

In order to get a better understanding of the origin of relations (32), we shall give a second method to solve Eqs. (14) which is a direct generalization of the formalism developed by Lemarchand.[13] From (10) we obtain $L=\Lambda^{-1}(-B^T)\Lambda$, where B is a diagonal matrix in the critical space of dimension n with basis vectors

$$\{\mathbf{e}_1=(1, 0,..., 0),..., \mathbf{e}_n=(0,..., 0, 1)\}$$

i.e., $B\mathbf{e}_j=s_j\mathbf{e}_j$ with $s_j=\mu_j$. One has $1\leqslant j\leqslant m$, and $s_{m+2k-1}=\mu_{m+k}+i\Omega_k$, $s_{m+2k}=\mu_{m+k}-i\Omega_k$, $1\leqslant k\leqslant p$. Putting $\theta=\Lambda^{-1}$, we see that the vectors $\{\theta\mathbf{e}_j, 1\leqslant j\leqslant n\}$ are eigenvectors of L with eigenvalues $(-s_j)$. Then any matrix C with columns proportional to the components θ_{jk} of the vectors $\theta\mathbf{e}_k$ will diagonalize L. We take for C the matrix

$$C_{jk}=-\frac{2\mu_k}{Q_{kk}}\theta_{jk}, \qquad 1\leqslant j\leqslant n, \quad 1\leqslant k\leqslant m \tag{34a}$$

$$\begin{aligned} C_{j,m+2k-1}&=-\frac{2\mu_{m+k}}{Q_{m+2k-1,m+2k}}\theta_{j,m+2k}, \qquad 1\leqslant k\leqslant p \\ C_{j,m+2k}&=-\frac{2\mu_{m+k}}{Q_{m+2k-1,m+2k}}\theta_{j,m+2k-1}, \qquad 1\leqslant k\leqslant p \end{aligned} \tag{34b}$$

and from (10) we see that $\theta_{jk}=-Q_{jk}/(s_j+s_k)$.

One has $(C^{-1}LC)_{jk}=r_j\delta_{jk}$, with $r_j=-\mu_j$; $1\leqslant j\leqslant m$; and $r_{m+2k-1}=-\mu_{m+k}+i\Omega_k$, $r_{m+2k}=-\mu_{m+k}-i\Omega_k$, $1\leqslant k\leqslant p$. We perform in (14) the change of variables $q_j=\sum_{k=1}^n C_{jk}Q_k$; then, putting

$$\tilde{\phi}^{[r]}(Q)=\phi^{[r]}(q), \qquad \tilde{I}^{[r]}(Q)=I^{[r]}(q)$$

we find that Eq. (14) becomes

$$\tilde{L}\tilde{\phi}^{[r]}(Q)\equiv\sum_{r=1}^{n} r_jQ_j\frac{\partial}{\partial Q_j}\tilde{\phi}^{[r]}(Q)=\tilde{I}^{[r]}(Q) \tag{35}$$

One has $C=D+O(\mu_j)$, with

$$D_{jk}=\frac{2\mu_k}{Q_{kk}}\frac{Q_{jk}}{\mu_j+\mu_k}, \qquad 1\leqslant j, k\leqslant m \tag{36}$$

and

$$D_{jk}=\delta_{jk} \qquad \text{if either} \quad j\geqslant m \text{ or } k\geqslant m$$

The change of variables is then of the form

$$q_j = Q_j + \sum_{\substack{k=1 \\ k \neq j}}^{m} \frac{Q_{jk}}{\mu_j + \mu_k} Q_k + O(\mu_k Q_l), \qquad 1 \leqslant j \leqslant m \tag{37a}$$

$$q_{m+k} = Q_{m+k} + O(\mu_j Q_l), \qquad 1 \leqslant k \leqslant 2p \tag{37b}$$

Putting

$$\tilde{\phi}^{[r]}(Q) = \sum_{j_k} \tilde{a}^{(r)}_{j_1 \cdots j_n} Q_1^{j_1} \cdots Q_n^{j_n}$$

$$\tilde{I}^{[r]}(Q) = \sum_{j_k} \tilde{b}^{(r)}_{j_1 \cdots j_n} Q_1^{j_1} \cdots Q_n^{j_n}, \qquad j_k \geqslant 0, \qquad \sum_{k=1}^{n} j_k = r$$

we obtain from (35)

$$\tilde{a}^{(r)}_{j_1 \cdots j_n} = \tilde{b}^{(r)}_{j_1 \cdots j_n} / \sum_{i=1}^{n} r_i j_i \tag{38}$$

one has

$$\sum_{i=1}^{n} r_i j_i = -\sum_{i=1}^{n} \mu_i j_i - \sum_{k=1}^{p} |\mu_{m+k}(j_{m+2k-1} + j_{m+2k}) + i\Omega_k (j_{m+2k} - j_{m+2k-1})| \tag{39}$$

which vanishes in the limit $\mu_j \to 0$ when $j_{m+2k} = j_{m+2k-1}$, i.e., just for the resonant terms

$$Q_1^{j_1} \cdots Q_m^{j_m} \cdot |Q_{m+1}|^{2j_{m+1}} \cdots |Q_{m+2p-1}|^{2j_{m+p}}$$

which are the elements of Ker D. We see then that in the variables $\{Q_j\}$ the potential will be singular in the space of parameters in all lines

$$\sum_{i=1}^{n} j_i \mu_i + 2 \sum_{k=1}^{p} \mu_{m+k} j_{m+k} = 0 \tag{40}$$

where $\{j_i\}$ are entire numbers positive or zero and $\sum_{i=1}^{n} j_i + 2 \sum_{k=1}^{p} j_{m+k} = r \geqslant 3$ [from (38) this holds whenever $\tilde{b}^{(r)}_{j_1 \cdots j_n} \neq 0$]. We can elimine these singularities, imposing in (38) that when $j_{m+2k-1} = j_{m+2k}$ one must have

$$\tilde{b}^{(r)}_{j_1 \cdots j_n} = \tilde{c}^{(r)}_{j_1 \cdots j_n} \cdot \sum r_i j_i$$

with $\tilde{c}^{(r)}_{j_1 \cdots j_n}$ nonsingular in the limit $\mu_j \to 0$ and these relations can be seen to be equivalent to the set of relations (32) previously found. We shall come back to this point in our discussion of some illustrative examples in the next section. We remark that if $m > 1$, one also has singularities of the potential in all lines $\mu_j + \mu_k = 0$, $j \leqslant m$, $k \leqslant m$ [see (35)] due to the change of variables. These singularities are eliminated if we impose $Q_{ij} = Q_{ii}\delta_{ij}$, $i \leqslant m$, $j \leqslant m$ [see after Eq. (12b), where these relations were also found] and these conditions are the first we have to impose on the coefficient of the master equation in order to have a polynomial approximation for the potential. If they are satisfied, then (37) shows that the change of variables reduces to $q_j = Q_j + O(\mu_k Q_l)$, the resonant terms are the same in both sets of variables, and $\phi^{[r]}(q) = \tilde{\phi}^{[r]}(Q = q)$ at lowest order in the $\{\mu_j\}$.

3. SOME ILLUSTRATIVE EXAMPLES

Codimension-one instabilities are simple to treat and no relations are needed to have a polynomial approximation[2,3] to the potential. We shall give here the results for the instabilities $(\xi\Omega)$, $(\Omega_1\Omega_2)$, of codimension two, and $(\xi\Omega_1\Omega_2)$, $(2\xi)(\Omega)$, of codimension three.

(a) The $(\xi\Omega)$ instability has one eigenvalue zero associated with a variable q_1 and a pair of pure imaginary complex conjugate eigenvalues $\pm i\Omega$ associated with the complex variable z $(q_2 = z, q_3 = \bar{z})$. With the symmetry $q_1 \to -q_1$ the normal form of $H_\mu(q)$ is

$$H_1(q) = \mu_1 q_1 + b_1 q_1^3 + b_2 q_1 |z|^2 \tag{41a}$$

$$H_2(q) = (\mu_2 + i\Omega) z + z(c_1 q_1^2 + c_2 |z|^2) \tag{41b}$$

and $H_3(q) = \overline{H_2(q)}$. The coefficients (b_1, b_2) are real and (c_1, c_2) complex. It has codimension two with unfolding parameters μ_1 and μ_2. From (33) we obtain (here $r' = 4$)

$$\begin{aligned}\phi = &-\frac{\mu_1}{Q_{11}} q_1^2 - 2\frac{\mu_2}{Q_{23}} |z|^2 + a_{0;40}^{(4)} q_1^4 \\ &+ a_{0;21}^{(4)} q_1^2 |z|^2 + a_{0;02}^{(4)} |z|^4\end{aligned} \tag{42}$$

with (Re stands for real part)

$$\begin{aligned} a_{0;40}^{(4)} &= -\frac{b_1}{2Q_{11}} \\ a_{0;21}^{(4)} &= -\frac{b_2}{Q_{11}} = -\frac{2\,\mathrm{Re}\, c_1}{Q_{23}} \\ a_{0;02}^{(4)} &= -\frac{\mathrm{Re}\, c_2}{Q_{23}} \end{aligned} \tag{43}$$

We see that one relation is needed in this case due to the double determination of $a^{(4)}_{0;21}$ in (43); it is $Q_{23}b_2 = 2Q_{11}\operatorname{Re} c_1$.

Using the second method, we obtain from formula (38) that $\tilde{a}^{(4)}_{400} = a^{(4)}_{0;40}$, $\tilde{a}^{(4)}_{022} = a^{(4)}_{0;02}$, and

$$\tilde{a}^{(4)}_{211} = -\frac{\mu_1(b_2/Q_{11}) + 2\mu_2(\operatorname{Re} c_1)/Q_{23}}{\mu_1+\mu_2} \tag{44}$$

We see then that if the relation $Q_{23}b_2 = 2Q_{11}\operatorname{Re} c_1$ is satisfied, the numerator in (42) is proportional to $(\mu_1+\mu_2)$ and one obtains $\tilde{a}^{(4)}_{211} = a^{(4)}_{0;21}$, in agreement with our general discussion after Eq. (40). As we explained there, this is the general mechanism at the origin of the relations needed to have polynomial expansions.

(b) The $(\Omega_1\Omega_2)$ instability has two pairs of complex conjugate pure imaginary eigenvalues $\pm i\Omega_1$ and $\pm i\Omega_2$ related, respectively, to the variables $q_1 = z_1$, $q_2 = \bar{z}_1$, $q_3 = z_2$, $q_4 = \bar{z}_2$. It has codimension 2 with unfolding parameters (μ_1, μ_2). The normal form of $H_\mu(q)$ is

$$H_1(q) = (\mu_1 + i\Omega_1)\, z_1 + z_1(d_1\,|z_1|^2 + d_2\,|z_2|^2) \tag{45a}$$

$$H_3(q) = (\mu_2 + i\Omega_2)\, z_2 + z_2(e_1\,|z_1|^2 + e_2\,|z_2|^2) \tag{45b}$$

with $H_2(q) = \overline{H_1(q)}$, $H_4(q) = \overline{H_3(q)}$, and where the coefficients (d_1, d_2, e_1, e_2) are complex. From (33) we obtain $(r' = 4)$

$$\begin{aligned}\phi = {} & -2\frac{\mu_1}{Q_{23}}\,|z_1|^2 - 2\frac{\mu_2}{Q_{34}}\,|z_2|^2 + a^{(4)}_{0;20}\,|z_1|^4 \\ & + a^{(4)}_{0;11}\,|z_1|^2\,|z_2|^2 + a^{(4)}_{0;02}\,|z_2|^4\end{aligned} \tag{46}$$

with

$$\begin{aligned} a^{(4)}_{0;20} &= -\frac{\operatorname{Re} d_1}{Q_{12}} \\ a^{(4)}_{0;11} &= -2\frac{\operatorname{Re} d_2}{Q_{12}} = -2\frac{\operatorname{Re} e_1}{Q_{34}} \\ a^{(4)}_{0;02} &= -\frac{\operatorname{Re} e_2}{Q_{34}} \end{aligned} \tag{47}$$

We need one relation here due to the double determination of $a^{(4)}_{0,11}$ and it is $Q_{12}\operatorname{Re} f = Q_{34}\operatorname{Re} d$. With the second method we obtain from (38) that

$$\tilde{a}^{(4)}_{2200} = a^{(4)}_{0;20}, \qquad \tilde{a}^{(4)}_{0022} = a^{(4)}_{0;02}$$

$$\tilde{a}^{(4)}_{1111} = -2\frac{\mu_1(\operatorname{Re} d_2)/Q_{12} + \mu_2(\operatorname{Re} e_1)/Q_{34}}{\mu_1+\mu_2} \tag{48}$$

which reduces to $a^{(4)}_{0;11}$ when the condition $Q_{12}\operatorname{Re} e_1 = Q_{34}\operatorname{Re} d_2$ is satisfied. This corroborates what we said in the previous example.

In the next examples we only give the results of the first method.

(c) The $(\xi\Omega_1\Omega_2)$ instability has one eigenvalue zero associated with a variable q_1, and two pairs of pure imaginary complex conjugate eigenvalues $\pm i\Omega_1$ associated with $(q_2 = z_1, q_3 = \bar{z}_1)$ and $\pm i\Omega_2$ associated with $(q_4 = z_2, q_5 = \bar{z}_2)$. With the symmetry $q_1 \to -q_1$ the normal form of $H_\mu(q)$ is

$$H_1(q) = \mu_1 q_1 + d_1 q_1^3 + e_1 q_1 |z_1|^2 + f_1 q_1 |z_2|^2 \tag{49a}$$

$$H_2(q) = (\mu_2 + i\Omega_1) z_1 + z_1(d_2 q_1^2 + e_2 |z_1|^2 + f_2 |z_2|^2) \tag{49b}$$

$$H_4(q) = (\mu_3 + i\Omega_2) z_2 + z_2(d_3 q_1^2 + e_3 |z_1|^2 + f_3 |z_2|^2) \tag{49c}$$

$$H_3(q) = \overline{H_2(q)}. \qquad H_5(q) = \overline{H_4(q)}$$

It has codimension 3 with unfolding parameters (μ_1, μ_2, μ_3), the coefficients (d_1, e_1, f_1) are real, and (d_j, e_j, f_j), $j \geqslant 2$, are complex. From (33) we obtain $(r' = 4)$

$$\begin{aligned} \phi = & -\frac{\mu_1}{Q_{11}} q_1^2 - 2\frac{\mu_2}{Q_{23}} |z_1|^2 - 2\frac{\mu_3}{Q_{45}} |z_2|^2 \\ & + a_{0;400}^{(4)} q_1^4 + a_{0;210}^{(4)} q_1^2 |z_1|^2 + a_{0;201}^{(4)} q_1^2 |z_2|^2 \\ & + a_{0;011}^{(4)} |z_1|^2 |z_2|^2 + a_{0;020}^{(4)} |z_1|^4 + a_{0;002}^{(4)} |z_2|^4 \end{aligned} \tag{50}$$

Following the arguments given after Eqs. (31) and (32), we see that each of three coefficients $(a_{0;210}^{(4)}, a_{0;201}^{(4)}, a_{0;011}^{(4)})$ will appear in two equations [$q = 2$ in (31)] and consequently will be at the origin of one relation. The other coefficients appear only in one equation [$q = 1$ in (31)], so that finally we shall have three relations in this case. One obtains

$$a_{0;400}^{(4)} = -\frac{d_1}{2Q_{11}}, \qquad a_{0;020}^{(4)} = -\frac{\operatorname{Re} e_2}{Q_{23}}, \qquad a_{0;002}^{(4)} = -\frac{\operatorname{Re} f_3}{Q_{45}} \tag{51a}$$

$$a_{0;210}^{(4)} = -\frac{e_1}{Q_{11}} = -\frac{2 \operatorname{Re} d_2}{Q_{23}} \tag{51b}$$

$$a_{0;201}^{(4)} = -\frac{f_1}{Q_{11}} = -\frac{2 \operatorname{Re} d_3}{Q_{45}} \tag{51c}$$

$$a_{0;011}^{(4)} = -\frac{2 \operatorname{Re} f_2}{Q_{23}} = -\frac{2 \operatorname{Re} e_3}{Q_{45}} \tag{51d}$$

The three relations will be $2Q_{11} \operatorname{Re} d_2 = e_1 Q_{23}$, $2Q_{11} \operatorname{Re} d_3 = f_1 Q_{45}$, $\operatorname{Re} e_3 Q_{23} = \operatorname{Re} f_2 Q_{45}$.

(d) The $(2\xi)(\Omega)$ instability has two eigenvalues zero (variables q_1 and q_2) and one pair of pure imaginary eigenvalues $\pm i\Omega$ (variables

$q_3 = z, q_4 = \bar{z}$). With the symmetry $q_1 \to -q_1$, $q_2 \to -q_2$ (both simultaneously), the normal form of $H_\mu(q)$ is

$$H_1(q) = \mu_1 q_1 + b_1 q_1^3 + b_2 q_1^2 q_2 + b_3 q_1 q_2^2 + b_4 q_2^3 + b_5 q_1 |z|^2 + b_6 q_2 |z|^2 \quad (52a)$$

$$H_2(q) = \mu_2 q_2 + c_1 q_2^3 + c_2 q_2^2 q_1 + c_3 q_2 q_1^2 + c_4 q_1^3 + c_5 q_2 |z|^2 + c_6 q_1 |z|^2 \quad (52b)$$

$$H_3(q) = (\mu_3 + i\Omega) z + z(d_1 |z|^2 + d_2 q_1^2 + d_3 q_1 q_2 + d_4 q_2^2) \quad (52c)$$

and $H_4(q) = \overline{H_3(q)}$. The coefficients $\{b_j, c_j\}$ are real and $\{d_j\}$ complex. It has codimension 3 with unfolding parameters (μ_1, μ_2, μ_3). In order to have a polynomial expansion in this case, one has to impose $Q_{12} = Q_{21} = 0$ [see after Eq. (12)]. Then from (33) one obtains

$$\begin{aligned}\phi = & -\frac{2\mu_1}{Q_1} q_1^2 - \frac{2\mu_2}{Q_{22}} q_2^2 - \frac{2\mu_3}{Q_{34}} |z|^2 + a^{(4)}_{0;400} q_1^4 \\ & + a^{(4)}_{0;310} q_1^3 q_2 + a^{(4)}_{0;220} q_1^2 q_2^2 + a^{(4)}_{0;130} q_1 q_2^3 \\ & + a^{(4)}_{0;040} q_2^4 + a^{(4)}_{0;201} q_1^2 |z|^2 + a^{(4)}_{0;111} q_1 q_2 |z|^2 \\ & + a^{(4)}_{0;021} q_2^2 |z|^2 + a^{(4)}_{0;002} |z|^4\end{aligned} \quad (53)$$

$$\begin{aligned} a^{(4)}_{0;400} &= -\frac{b_1}{2Q_{11}} \\ a^{(4)}_{0;040} &= -\frac{c_1}{2Q_{22}} \\ a^{(4)}_{0;002} &= -\frac{\mathrm{Re}\, d_1}{Q_{34}} \\ a^{(4)}_{0;310} &= -\frac{2b_2}{3Q_{11}} = -\frac{2c_4}{Q_{22}} \\ a^{(4)}_{0;220} &= -\frac{b_3}{Q_{11}} = -\frac{c_3}{Q_{22}} \\ a^{(4)}_{0;130} &= -\frac{2b_4}{Q_{11}} = -\frac{2c_2}{3Q_{22}} \\ a^{(4)}_{0;201} &= -\frac{b_5}{Q_{11}} = -\frac{2\,\mathrm{Re}\, d_2}{Q_{34}} \\ a^{(4)}_{0;111} &= -\frac{2b_6}{Q_{11}} = -\frac{2c_6}{Q_{22}} = -\frac{2\,\mathrm{Re}\, d_3}{Q_{34}} \\ a^{(4)}_{0;021} &= -\frac{c_5}{Q_{22}} = -\frac{2\,\mathrm{Re}\, d_4}{Q_{34}} \end{aligned} \quad (54)$$

We have here seven relations,

$$\frac{b_3}{Q_{11}}=\frac{c_3}{Q_{22}}, \qquad \frac{b_5}{Q_{11}}=\frac{2\,\mathrm{Re}\,d_2}{Q_{34}}, \qquad \frac{c_5}{Q_{22}}=\frac{2\,\mathrm{Re}\,d_4}{Q_{34}} \tag{55a}$$

$$\frac{b_2}{3Q_{11}}=\frac{c_4}{Q_{22}}, \qquad \frac{b_4}{Q_{11}}=\frac{c_2}{3Q_{22}}, \qquad \frac{b_6}{Q_{11}}=\frac{\mathrm{Re}\,d_3}{Q_{34}}=\frac{c_6}{Q_{22}} \tag{55b}$$

We remark that if we impose the symmetry $q_1 \to -q_1$ and independently $q_2 \to -q_2$, we have that $b_2=b_4=b_6=c_2=c_4=c_6=d_3=0$ and the second set of relations (55b) is automatically satisfied.

To close this section, we give a practical method to obtain the polynomial potential and the relations which guarantee its existence. From the general discussion one knows that $\phi=\phi^{[2]}+\phi_0^{[r']}$ [see (33)]; then one computes

$$H'_j(q)=-\frac{1}{2}Q_{jj}\frac{\partial\phi}{\partial q_j}, \qquad 1\leqslant j\leqslant m$$

$$H'_{m+2k-1}=-\frac{1}{2}Q_{m+2k-1,m+2k}\frac{\partial\phi}{\partial\bar{z}_k}, \qquad 1\leqslant k\leqslant p$$

and identifies $H'_\alpha(q)$ with $H''_\alpha(q)$, where $H''_\alpha(q)$ is obtained from $H_\alpha(q)$ [given by (8)], replacing there all the coefficients of the monomials by their real part. This determines all the unknown coefficients in $\phi_0^{[r']}$ and also the relations when one of the coefficients is determined more than once. These rules mean that in the case of multiple Hopf bifurcations $(\Omega_1, \Omega_2, ..., \Omega_p)$ the potential depends only on the real part of the coefficients of the deterministic equation and consequently the attractors and repulsors near the bifurcation are determined only by these coefficients.

4. CONCLUSIONS

We have considered here the general problem of obtaining polynomial expansions in the gross variables and in the unfolding parameters for the nonequilibrium potential associated with a general master equation in the neighborhood of an instability with diagonalizable linear part. We have shown that these expansions exist, provided a set of relations is satisfied among the coefficients of the master equation. But at the same time these results are also relevant for the study of dynamical systems near an instability. Consider the dynamical system $\dot{q}_\mu=H_\mu(q)$ and let $q(t)$ be solution. Then if $Q_{\mu\nu}$ is a positive-definite matrix and if $\phi(q)$ is a solution of the Hamilton–Jacobi equation $H(q,\nabla\phi)=0$ with the Hamiltonian

$$H(q,p)=\sum_{\mu,\nu} p_\mu(H_\mu(q)+\tfrac{1}{2}Q_{\mu\nu}p_\nu) \tag{56}$$

we have

$$\frac{d}{dt}\phi(q(t)) = -\frac{1}{2}\sum_{\mu,\nu} Q_{\mu\nu}\frac{\partial\phi}{\partial q_\mu}\frac{\partial\phi}{\partial q_\nu} \leqslant 0 \tag{57}$$

i.e., $\phi(q)$ is a generalized Lyapunov functional.[1,14] We can interpret this by introducing the white noises $\{\xi_\mu(t)\}$ with zero mean and correlations $\langle \xi_\mu(t)\,\xi_\nu(t')\rangle = Q_{\mu\nu}\delta(t-t')$ and considering the stochastic differential equations

$$\dot{q}_\mu = H_\mu(q) + \eta^{1/2}\xi_\mu(t) \tag{58}$$

The Fokker–Planck equation associated with (58) will be

$$\frac{\partial}{\partial t}p(q,t) = \sum_{\mu,\nu}\frac{\partial}{\partial q_\mu}\left[-H_\mu(q) + \frac{\eta}{2}Q_{\mu\nu}\frac{\partial}{\partial q_\nu}\right]p(q,t) \tag{59}$$

which is of the form (2), and the associated classical Hamiltonian in the sense of (5) will be (56). In the weak noise limit ($\eta \to 0$) the stationary probability associated with (59) will be $\exp[-(1/\eta)\,\phi(q)]$ and $\phi(q)$ will be Lyapunov functional in the sense of (57) for the dynamical system $\dot{q}_\mu = H_\mu(q)$. From this point of view we should also remark that for lower codimensions the relations which must hold in order to have polynomial expansions do not fix the values of the coefficients of $H_\mu(q)$, but only impose inequalities among them (in fact, for codimension-one instabilities one finds no relations[2]). The reason for this is that if our problem is to find a Lyapunov functional for $\dot{q}_\mu = H_\mu(q)$, the matrix $Q_{\mu\nu}$ is arbitrary apart from the fact that it must be positive definite. We can illustrate this with the $(\xi\Omega)$ instability (example a) of Section 3, where the relation is $Q_{23}b_2 = 2Q_{11}\operatorname{Re} c_1$, which only imposes $b_2 \operatorname{Re} c_1 > 0$, since Q_{11} and $Q_{23} = Q'_{11} + Q'_{33}$ are positive ($Q'_{\alpha\beta}$ is the matrix corresponding to the real variables, which must be positive definite; see the last paragraph of Section 1). The same holds for examples b and c of Section 3. In example d, which is of codimension three, the situation is different: relations (55a) give again only inequalities between $(b_3, b_5, c_3, c_5, \operatorname{Re} d_2, \operatorname{Re} d_4)$ and fix (Q_{11}, Q_{22}, Q_{34}), but then the four relations (55b) determine $(c_2, c_4, c_6, \operatorname{Re} d_3)$ in terms of (b_2, b_4, b_6). This is in fact the generic case for higher codimensions. It should be remarked here that many important features of the deterministic behavior of the dynamical system $\dot{q}_\mu = H_\mu(q)$ near the instability can be obtained from the polynomial potential when it exists.[2,4,12] In this context we must mention that the existence of a polynomial potential does not exclude the possibility of having limit cycles or coexisting attractors. A simple example is the dynamical system

$$\dot{z} = (\mu + i\Omega)\,z + (1 + i\beta)\,z\,|z|^2 - (1 + i\delta)\,z\,|z|^4 + \eta^{1/2}\xi(t) \tag{60}$$

where $z(t)$ is a complex variable, $\xi = \xi_1 + i\xi_2$, with (ξ_1, ξ_2) white noises with zero mean and correlations $\langle \xi_j(t)\, \xi_k(t') \rangle = Q\delta_{jk}\, \delta(t-t')$. The potential here is

$$\phi = -\frac{1}{Q}\left(\mu\, |z|^2 + \frac{1}{2}\, |z|^4 - \frac{1}{3}\, |z|^6\right) \tag{61}$$

and for $-1/4 < \mu < 0$ one has coexistence of the stable fixed point $z = 0$ with the stable limit cycle $|z| = (1/\sqrt{2})[1 + (1 + 4\mu)^{1/2}]^{1/2}$. The question is open of what happens when polynomial expansions do not exist, although a natural conjecture is that chaotic behavior is not compaticle with a polynomial potential. One reason for the failure of a polynomial expansion can be found in formulas (44) and (48), which exhibit singular denominators. However, here we have looked for polynomials in a special set of variables q and this does not exclude good polynomial approximations in other variables. This is indeed the case for the ξ^2 instability.[17]

The techniques preseted here can be used to study spatially extended systems; however, the appearance of bands of critical modes introduces new conditions for the existence of polynomial potentials.[15] In the case when the linear part of the field $H_\mu(q)$ is not diagonalizable and presents Jordan blocks, one has a different situation, which will be discussed elsewhere.

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